Advance KENTUCKY

AP Calculus Prep Session Handout

Integral-Defined Functions

Integral-Defined Functions

A continuous, differentiable function can be expressed as a definite integral if it is difficult or impossible to determine the antiderivative of a function using known integration methods. Using the antiderivative form of the Fundamental Theorem of Calculus, it is possible to learn about the function without expressing it in a closed form.

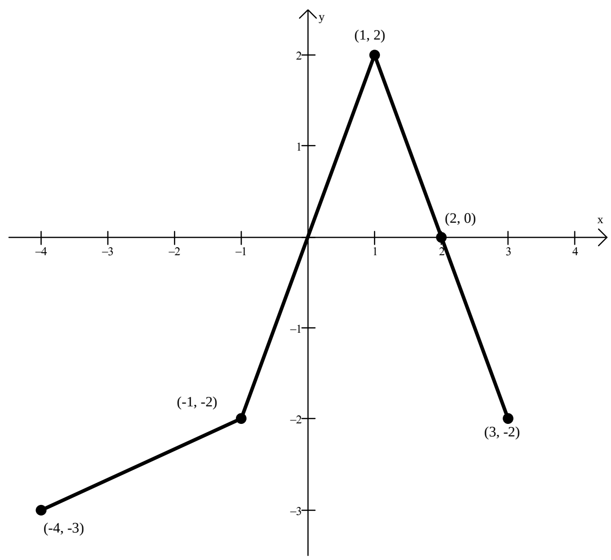
Functions expressed as definite integrals appear often on the AP Calculus examinations in both free response and multiple choice questions. These questions often involve the graph of the integrand of the definite integral and expect students to apply the Fundamental Theorem of Calculus and derivative properties and theorems to the integral-defined functions. Occasionally students are asked to write an expression for a function that can only be expressed using a definite integral in application problems.

What students should be able to do:

- Evaluate a function defined by a definite integral at a given value of x.
- Apply the Fundamental Theorem of Calculus to a function defined by a definite integral. That is, $\frac{d}{dx}\left(\int_0^x f(t)dt\right) = f(x)$. If the upper limit of the integral is a function, then $\frac{d}{dx}\left(\int_0^{g(x)} f(t)dt\right) = f\big(g(x)\big)g'(x)$.
- To determine for what values of x is a function defined by an integral is increasing or decreasing.
- To determine for what values of x is the graph of a function defined by an integral is concave up or concave down.
- To find relative and absolute extrema and points of inflection of functions defined by integrals.
- Write an integral expression involving a definite integral for a function given its derivative.

Free Response Questions

2005 AB 4 BC 4 Form B



Graph of *f*

The graph of the function *f* above consists of three line segments.

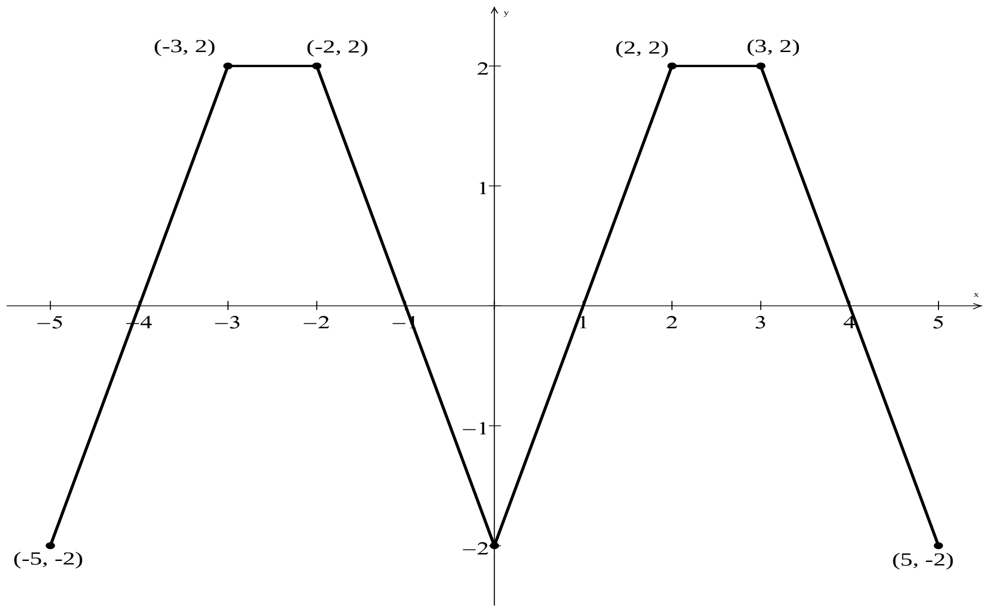
(a) Let *g* be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$. For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function *g* defined in part (a), find the *x*-coordinate of each point of inflection of the graph of *g* on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let *h* be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of *x* in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function *h* defined in part (c), find all intervals on which *h* is decreasing. Explain your reasoning.

2006 AB 3



Graph of f

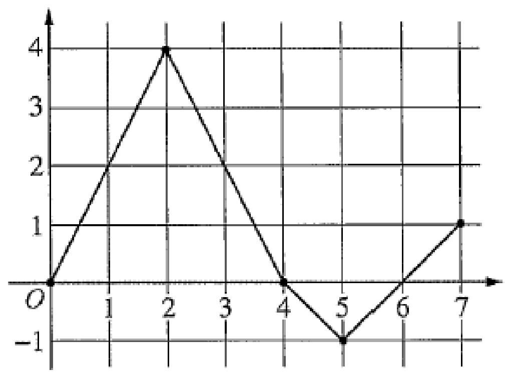
The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)dt$.

(a) Find $g(4), g'(4)$, and $g''(4)$.

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

2003 AB 5 and BC 5 Form B



Graph of f

Let f be a function defined on the closed interval [0, 7]. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

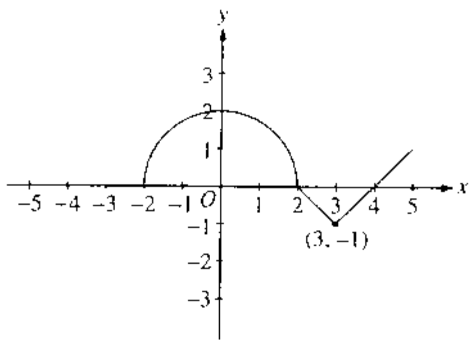
(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values of c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning,

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

1997 AB 5 and BC 5



The graph of a function *f* consists of a semicircle and two line segments as shown above. Let *g* be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(3)$.

(b) Find all values of *x* on the open interval $(-2, 5)$ at which *g* has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of *g* at $x = 3$.

(d) Find the *x*-coordinate of each point of inflection of the graph of *g* on the open interval $(-2, 5)$. Justify your answer.

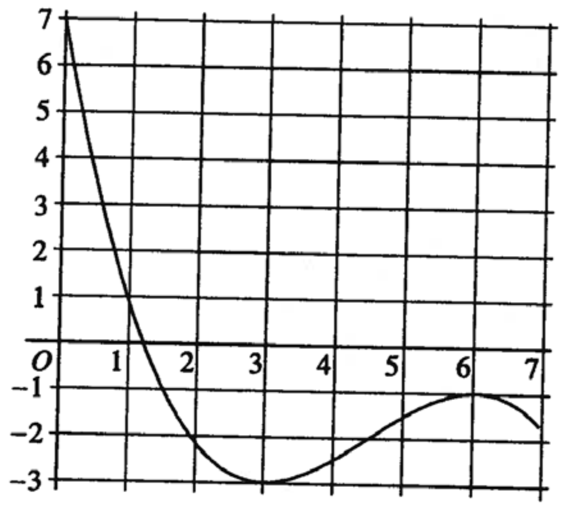
Multiple Choice Questions

Part A. No Calculator.

1. $\frac{d}{dx}\left(\int_0^{x^2} \sin(t^3)\ dt\right) =$
   (A) $-\cos(x^6)$ (B) $\sin(x^3)$ (C) $\sin(x^6)$ (D) $2x\sin(x^3)$ (E) $2x\sin(x^6)$

2. $\frac{d}{dx}\left(\int_0^{x^3} \ln(t^2+1)\ dt\right) =$
   (A) $\frac{2x^3}{x^6+1}$ (B) $\frac{3x^2}{x^6+1}$ (C) $\ln(x^6+1)$ (D) $2x^3\ln(x^6+1)$ (E) $3x^2\ln(x^6+1)$

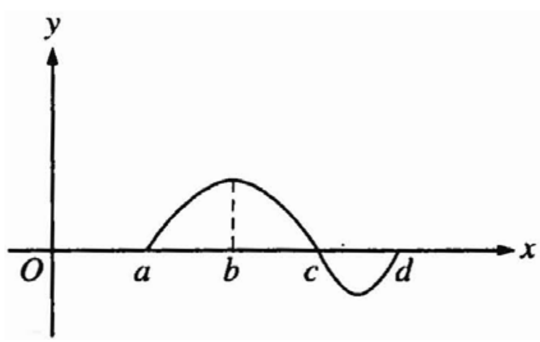
3. If $F(x) = \int_0^x \sqrt{t^3+1}\, dt$, then $F'(2) =$
   (A) -3 (B) -2 (C) 2 (D) 3 (E) 18



Graph of f

4. The graph of the function f shown in the figure above has horizontal tangents at $x = 3$ and $x = 6$. If $g(x) = \int_0^{2x} f(t)\,dt$, what is the value of $g'(3)$?

(A) 0 (B) -1 (C) -2 (D) -3 (E) -6



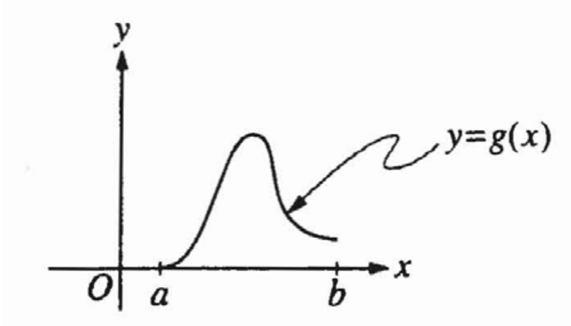
5. The graph of f is shown in the figure above. If $g(x) = \int_a^{x} f(t)\,dt$, for what value of x does $g(x)$ have a maximum?

(A) a
(B) b
(C) c
(D) d
(E) It cannot be determined from the information given.

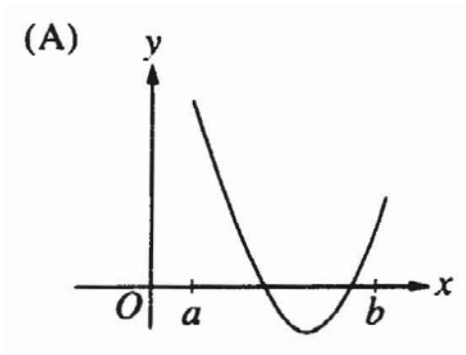
Part B. Graphing Calculator Allowed.

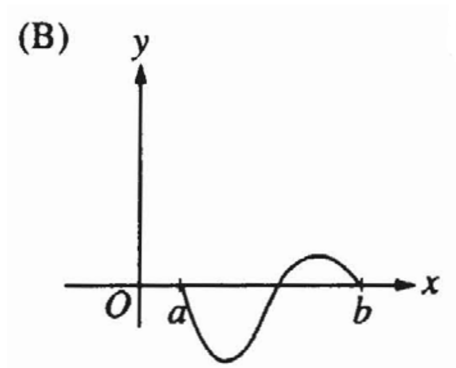
6. Let g be the function given by $g(x) = \int_0^x \sin(t^2)\ dt$ for $-1 \leq x \leq 3$. On which of the following intervals is g decreasing?
   (A) $-1 \leq x \leq 0$
   (B) $0 \leq x \leq 1.772$
   (C) $1.253 \leq x \leq 2.171$
   (D) $1.772 \leq x \leq 2.507$
   (E) $2.802 \leq x \leq 3$

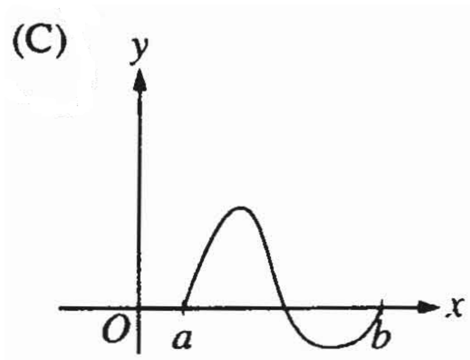
7. Let $f(x) = \int_0^{x^2} \sin t\ dt$. At how many points in the closed interval $\left[0, \sqrt{\pi}\right]$ does the instantaneous rate of change of f equal the average rate of change of f on that interval?
   (A) Zero
   (B) One
   (C) Two
   (D) Three
   (E) Four

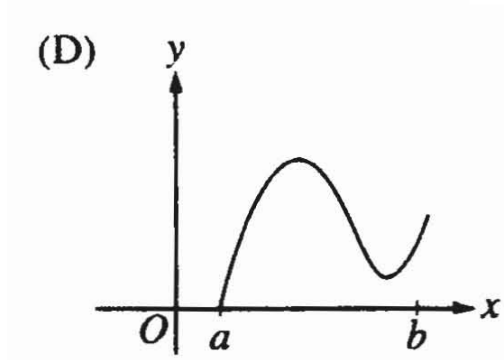


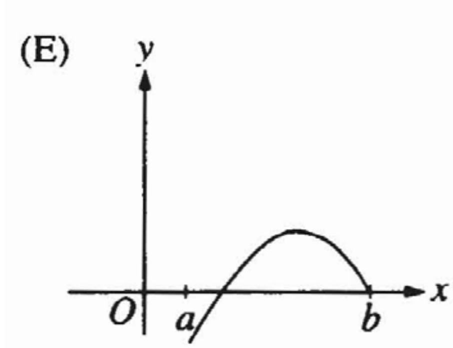
8. Let $g(x) = \int_a^x f(t)\,dt$, where $a \le x \le b$. The figure above shows the graph of g on [a, b]. Which of the following could be the graph of f on [a, b]?

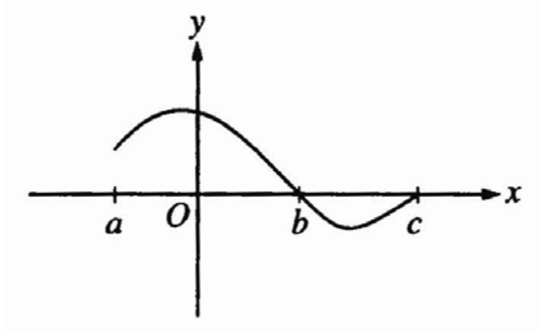






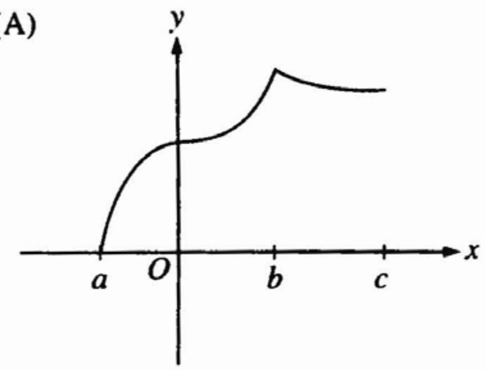




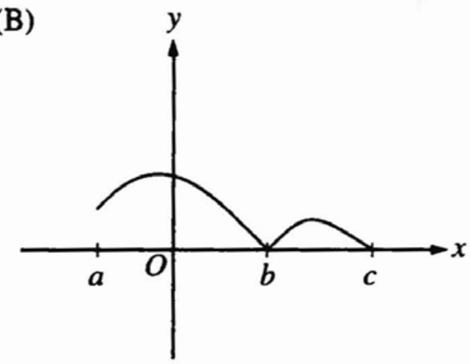


9. Let $f(x) = \int_a^x h(t)\,dt$ where h has the graph shown above. Which of the following could be the graph of f?

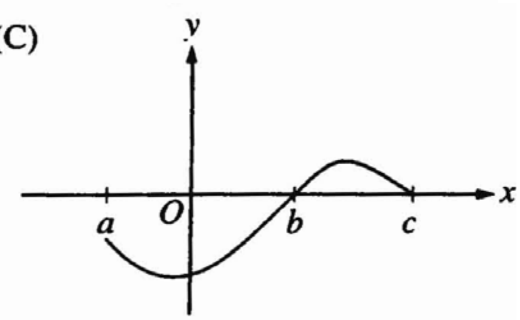
(A)



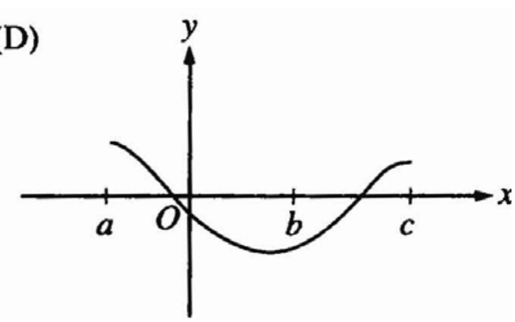
(B)



(C)



(D)



(E)